

A Method for Justification of the View of Observables in Quantum Mechanics and Probability Distributions in Phase Space

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Abstract

Let $f(x, p)$ be a function on the phase space. The function f corresponds to the classical observation. Let $\rho(x, p, \xi) = \varphi^2(x, p, \xi)$ be some non-negative density function on an extended phase space. The density function ρ corresponds to the generalized state on the extended phase space. Then, as usual, the observation f in the generalized state ρ is the value $\langle f, \rho \rangle = \langle f, \varphi^2 \rangle$ of the integral of the function f in the distribution ρ . It is supposed, that in quantum observations all distributions ρ are realized not, and only distributions $\rho = \tilde{\varphi}^2$, where $\tilde{\varphi}$ belongs to some linear subspace $\tilde{\mathcal{H}}$ (averaging wave functions) in space of all functions on the extended phase space. Besides it is supposed, that in quantum experiments values of the spectrum of the linear operator A_f of the quadratic form $\langle f, \tilde{\varphi}^2 \rangle = \langle \tilde{\varphi}, A_f \tilde{\varphi}^2 \rangle$, where $\tilde{\varphi} \in \tilde{\mathcal{H}}$, are observed. In this paper we consider certain hypotheses on the averaging process of wave functions. Then it is shown that the spectrum of usual operator of the quantum observable corresponded to f is small differing from the spectrum of the operator A_f . Other side, this approach enables one to recover probability distributions in the phase space for wave functions.

Introduction

In this paper there is considered a mathematical model of microworld processes based on certain constructions of the geometric quantization [K] which takes in account the fluctuating affects of the media, such as fluctuations of the vacuum. We show that an enlargement of the phase space and of its motion group and an account for the diffusion motions of microsystems in the enlarged space, the motions which act by small random translations along the enlarged group, lead to observable quantum effects.

I show that certain conventional nonrelativistic descriptions of quantum systems can be obtained as asymptotic approximations to the model suggested in this paper with respect to the power series expansion in Planck's constant. The parameters of the proposed model are estimated on the base of the experimental data of Lamb's shifts in the spectrum of hydrogen's atom [LR]. In quantum mechanics these shifts can not be explained in the framework of the nonrelativistic quantum model and the idea of vacuum's polarization around a point source (radiational corrections) is incorporated to do the trick.

On the one hand, the suggested model can be used to deepen our perception on the nature of quantum effects; on the other hand, it leads to yet another construction of quantizations of mechanical systems, the construction that might be of help in some exceptional cases.

Here I propose to consider probability amplitudes on the enlarged phase space which includes the space of inner states of the system, its own in each experiment. I assume that in the experiments on quantum systems our classical gadgets can only register the probability distribution corresponding to the averaged probability amplitudes over fluctuating actions on the quantum system. I will show that in the suggested model the averaged probability amplitudes on the phase space are parametrized by complex functions on the configuration space, i.e. the wave functions. In particular, formula (8) of Theorem 4.1 obtained in this model reflects the probability distribution for a wave function in the phase space. Wigner [Wi] who obtained so-called "quasidistributions" in the phase space was first solving this problem. However in certain cases his quasidistributions can be negative (unlike the ones obtained here) and therefore have no physical meaning.

In sec. 1 the main hypotheses on quantum observations are formulated and a formulation of the main problem is given.

In sec. 2 the action of the enlarged group of motions on the enlarged phase space is described.

In sec. 3 there are introduced hypotheses on the averaging operation and the theorem on the form of averaged distributions in the enlarged phase space is formulated.

In sec. 4 the main result of the paper is proved: the asymptotic proximity of linear operators of the considered model to the linear operators of quantum observables and the probability distribution in the phase space for a wave function.

In sec. 5 there are listed several unsolved problems and trends of further investigations.

Appendix contains proof of the theorem on the form of averaged distributions in the enlarged phase space and an estimate of the model's parameters based on the experimental data of Lamb's shift in the spectrum of hydrogen's atom.

1 Main hypotheses and formulation of the problem

First let us recall the main notions of classical mechanics for the flat configuration space, cf. e.g. [DFN, B2]. Let M (locally, R^{2n}) be the phase space with coordinates of its points being (q, p) , where $q = (q_1, \dots, q_n)$ is the point's position in the configuration space and $p = (p_1, \dots, p_n)$ is its momentum. In classical mechanics the dynamics is given by the formula $df/dt = \{f, H\}$, where $\{.,.\}$ is the Poisson bracket determined by the exterior 2-form ω (locally of the form $\sum_{1 \leq i \leq n} dq_i \wedge dp_i$) and H is a fixed function, the Hamiltonian.

The *canonical transformations* g of the phase space M are (smooth, 1-1) maps $g : M \rightarrow M$ preserving ω . Let S denote the group of all canonical transformations (*symplectic* diffeomorphisms) of the phase space. The dynamics of a mechanical system is given by a 1-parameter subgroup of S . To observables of a mechanical system the functions f on the phase space are assigned and to the states there are assigned nonnegative distributions $\rho dq dp$, or shortly, ρ . The mean value of an observable f in the state ρ is by definition

$$\langle f, \rho \rangle = \int_M f(q, p) \rho(q, p) dq dp.$$

We will make the following hypotheses on a quantum system on the above phase space:

1. *The phase space is enlarged to the space E which is the total space of a fibration $\pi : E \rightarrow M$ over M with fiber F which is a manifold of 'inner states' of the system (if $M = R^{2n}$ then $E = R^{2n} \times F$).*
2. *Let $pr : P \rightarrow S$ be the nontrivial central extension of the group S with the help of circle $T = R/hZ$ (Lie algebra of P is the Poisson algebra). The group P acts by diffeomorphisms on E so that $\pi \circ g' = pr(g') \circ \pi$ for any $g' \in P$ where $g' : E \rightarrow E$.*
3. *The observables of a quantum system, as well as of a classical one, are given by functions on the phase space.*

Denote by $\rho(q, p, \xi)dqdpd\xi$ a nonnegative distribution on E determined by a distribution density $\rho(q, p, \xi)$ with $\xi \in F$ and a P -invariant measure $dqdpd\xi$ on E . Let us present ρ in the form $\rho(q, p, \xi) = \varphi^2(q, p, \xi)$, where $\varphi(q, p, \xi)$ is a function on E such that

$$\int_F \varphi(q, p, \xi)d\xi = 0.$$

Denote by $\tilde{\rho}(q, p, \xi) = \tilde{\varphi}^2(q, p, \xi)$ an averaged density distribution and $\tilde{\varphi}(q, p, \xi)$ obtained from $\varphi(q, p, \xi)$ by an averaging process under the action of small fluctuations of the system on E (a mathematical model of this averaging process will be given in sec. 3).

4'. *Only averaged density distributions of the form $\tilde{\rho}(q, p, \xi) = \tilde{\varphi}^2(q, p, \xi)$ are realized in quantum measurements. The map $\varphi \mapsto \tilde{\varphi}$ is a linear projection operator whose form is given in sec.3, where the refined hypothesis 4), is given.*

5. *Only averaged values of observables f of the form*

$$\langle f, \tilde{\rho} \rangle = \int_E f(q, p) \tilde{\rho}(q, p, \xi) dqdpd\xi = \int_E f(q, p) \tilde{\varphi}^2(q, p, \xi) dqdpd\xi$$

are realised in measurements, where $\tilde{\rho}(q, p, \xi) = \tilde{\varphi}^2(q, p, \xi)$ and $\tilde{\varphi}$ is the averaging function on E .

Now, consider the completed Hilbert space $\tilde{\mathcal{H}} \subset L_2(E, dqdpd\xi)$ of averaged functions $\tilde{\varphi}(q, p, \xi)$ on E with respect to the standard inner product

$$\langle \tilde{\varphi}', \tilde{\varphi}'' \rangle = \int_E \tilde{\varphi}' \tilde{\varphi}'' dqdpd\xi.$$

To a function f on the phase space assign the linear operator A_f in $\tilde{\mathcal{H}}$ given by the formula

$$\langle \tilde{\varphi}, A_f \tilde{\varphi} \rangle \stackrel{def}{=} \langle f, \tilde{\rho} \rangle = \langle f, \tilde{\varphi}^2 \rangle.$$

The main result of this paper is the proof of the fact that under the natural P -action on E the operators A_f are approximately (up to terms of order \hbar , where \hbar is Planck's constant) coincide with the operators of quantum observables in the accept definition of quantum mechanics.

2 A description of the group P and its action on the enlarged space of states

Let $pr : P \longrightarrow S$ be the nontrivial central extension of the group S of canonical transformations with the help of the circle (1-dimensional torus) $T = R/\hbar Z$. Let $\mathfrak{po}(2n)$ be the Poisson algebra, the nontrivial central extension of the Lie algebra $\mathfrak{h}(2n)$ of Hamiltonian vector fields. In a sense that Lie theory is applicable to infinite dimensional case $\mathfrak{po}(2n)$ and $\mathfrak{h}(2n)$ are the Lie algebras of the groups P and S , respectively.

Two bundles E, E' over a symplectic manifold M (with fibers F, F' , respectively) with P -actions on them are equivalent if there is a diffeomorphism of bundles with P -action compatible with projections and the S -action on the base. To describe bundles with P -action over M we have to consider first the structure of the fiber F of the bundle E in detail.

Denote by S_0 the subgroup of the canonical transformations S that preserve the origin, let $P_0 = pr^{-1}(S_0)$. Clearly, $P_0 = T \oplus S_0$.

Since $P_0 \subset P$ acts on E and preserves the fiber F over the origin, a P -action in F is defined. In particular, $T \subset P$ acts in F .

Theorem 2.1. ([K]). *A bundle $E = R^{2n} \times F$ with a P -action compatible with an $\mathfrak{h}(2n)$ -action on the space of functions on R^{2n} is uniquely up to equivalence of P -bundles determined by the space F with a P_0 -action.*

To a P -action on E we assign in the standard way a Lie algebra homomorphism $\mathfrak{po}(2n) \longrightarrow \mathbf{vect}(E)$, the derivative of the P -action on E . Denote by τ the vector field on F corresponding to the action of the 1-parameter group generated by T on E (τ can also be defined as the value of the derivative of the action $T = T_t$ with respect to the parameter t at $t = 0$).

Corollary 2.2. *In any P -bundle E one can chose a trivialization (isomorphic to $R^{2n} \times F$) so that the vector fields on E corresponding to Hamiltonians linear in q and p , i.e. of the form $H = \sum_{i=1}^n (x_i p_i - y_i q_i) + c \in \mathfrak{po}(2n)$, are given in local coordinates (q, p, ξ) on E by the expression*

$$D_H = \sum_{i=1}^n (x_i \partial q_i + y_i \partial p_i) + (c - \sum_{i=1}^n y_i q_i) \tau.$$

Equivalently, in the global form, this means that for an arbitrary function $\varphi(q, p, \xi)$ on E the action of the one-parametric subgroups G_t^H for the above $H \in \mathfrak{po}(2n)$ is given by the formula

$$G_t^H \varphi(q, p, \xi) = \varphi(q + tx, p + ty, T_{t(c - \langle y, q \rangle)}(\xi)),$$

where $\langle y, q \rangle = \sum_{i=1}^n y_i q_i$ and $T_{t(c - \langle y, q \rangle)}$ is the element of the 1-parameter group T corresponding to the value $t(c - \langle y, q \rangle)$ of the parameter.

In what follows we set:

$$W_t^{x,y} = G_t^H \quad \text{for } H = \sum_{i=1}^n (x_i p_i - y_i q_i).$$

Denote by W the subgroup of P generated by $W_t^{x,y}$ for $(x, y) \in R^{2n}$ (usually W is called Heisenberg-Weyl group). Clearly, W is the inverse image of R^{2n} with respect to the projection $pr : P \rightarrow S$, where $R^{2n} \subset S$ is considered as the subgroup of translations.

3 Averaging

Now, let us pass to averaging of a function $\varphi(q, p, \xi)$ on a P -bundle E . We need it in hypothesis 4.

Let us start with the assumption that the averaging $\varphi \mapsto \tilde{\varphi}$ is associated with a diffusion process (Brownian motion) on a P -bundle caused by inaccuracy of the setting of the quantum observational device. This process acts locally by small shifts by Δq and Δp along the coordinates q and p of R^{2n} , respectively, and globally with the help of $W_1^{\Delta q, \Delta p} \in W$ acting on E .

More precisely, let τ be the diffusion time. The functions are transformed as for a diffusion process, cf. e.g. [I]:

$$\varphi(\tau + \Delta\tau, q, p, \xi) = \int_{R^{2n}} K(\Delta q, \Delta p, \Delta\tau) \times$$

$$\times W_1^{\Delta q, \Delta p} \varphi(\tau, q, p, \xi) d(\Delta q) d(\Delta p) + o(\Delta \tau), \quad (1)$$

where $K(\Delta q, \Delta p, \Delta \tau)$ is the probability density of shifts by the vector $(\Delta q, \Delta p)$ during the time $\Delta \tau$.

Naturally, we make the usual assumptions about $K(\Delta q, \Delta p, \Delta \tau)$:

- $K(\Delta q, \Delta p, \Delta \tau)$ is rapidly decreasing at infinity;
- the mathematical expectation of the shift vector $(\Delta q, \Delta p)$ is zero;
- the diagonal elements of the matrix of 2nd moments are of the form

$$\int_{R^{2n}} (\Delta q_i)^2 K(\Delta q, \Delta p, \Delta \tau) d(\Delta q) d(\Delta p) = 2a_i^2 \Delta \tau + o(\Delta \tau),$$

$$\int_{R^{2n}} (\Delta p_i)^2 K(\Delta q, \Delta p, \Delta \tau) d(\Delta q) d(\Delta p) = 2b_i^2 \Delta \tau + o(\Delta \tau),$$

where a and b characterize the "intensity" of shifts along positions and momentum;

- shifts along distinct directions are poorly correlated with each other, i.e. the offdiagonal 2-nd moments are of order $o(\Delta \tau)$;
- the moments of $K(\Delta q, \Delta p, \Delta \tau)$ of orders greater than 2 are also of order $o(\Delta \tau)$.

Starting from (1) and the above assumptions we expand $\varphi(\tau, q, p, \xi)$ in the Taylor series to derive as $\Delta \tau \rightarrow 0$ a differential equation similar to the diffusion equation. The equation represents the averaging of the function φ over fluctuations. The asymptotic of solutions of this equation as $\tau \rightarrow \infty$ is given by the following theorem, where j is the imaginary unit, $*$ denotes the complex conjugation and $\psi(x, \xi)$ is a complex-valued function on $R^n \times F$ such that

$$\psi(x, T_t(\xi)) = \psi(x, \xi) \exp \left(-j \frac{2\pi}{h} t \right). \quad (2)$$

Theorem 3.1. *Let $\varphi(\tau, q, p, \xi)$ satisfy (1) and $\varphi(0, q, p, \xi) = \varphi(q, p, \xi)$, where $\int_0^h \varphi(q, p, T_t(\xi)) dt = 0$. Then $\varphi(\tau, q, p, \xi)$ asymptotically approximates, as $\tau \rightarrow \infty$, to the function*

$$\tilde{\varphi}(q, p, \xi) \exp \left(-\tau \sum_{i=1}^n \frac{2\pi a_i b_i}{h} \right), \quad (3)$$

where

$$\tilde{\varphi}(q, p, \xi) = \frac{1}{\sqrt{2}} \left(\frac{2}{h^3} \right)^{\frac{n}{4}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^{\frac{1}{4}} \int_{R^n} \exp \left(-\frac{\pi}{h} \sum_{i=1}^n \frac{b_i}{a_i} (q_i - x_i)^2 \right) \times$$

$$\times \left(\psi(x, \xi) \exp\left(-j \frac{2\pi \langle p, x \rangle}{h}\right) + \psi^*(x, \xi) \exp\left(j \frac{2\pi \langle p, x \rangle}{h}\right) \right) dx. \quad (4)$$

The function $\psi(x, \xi)$ is obtained from the function $\varphi(q, p, \xi)$ according to the formula

$$\begin{aligned} \psi(x, \xi) = & \sqrt{2} \left(\frac{1}{h}\right) \left(\frac{2}{h^3}\right)^{\frac{n}{4}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n}\right)^{\frac{1}{4}} \int_0^h \int_{R^{2n}} \varphi(q, p, T_t(\xi)) \exp\left(j \frac{2\pi t}{h}\right) \times \\ & \times \exp\left(-\frac{\pi}{h} \sum_{i=1}^n \frac{b_i}{a_i} (q_i - x_i)^2\right) \exp\left(j \frac{2\pi \langle p, x \rangle}{h}\right) dq dp dt. \end{aligned} \quad (5)$$

Besides, if $\psi(x, \xi)$ is an arbitrary complex-valued function satisfying (2) then the composition of transformations $\psi \mapsto \tilde{\varphi}$ and $\tilde{\varphi} \mapsto \psi$ given by (4) and (5) is identity.

Proof is given in Appendix.

Now we are ready to refine hypothesis 4' from sec. 2 as follows:

4. Let $\varphi(q, p, \xi)$ be a real-valued function on E such that

$$\int_0^h \varphi(q, p, T_t(\xi)) dt = 0$$

and $\rho(q, p, \xi) = \varphi^2(q, p, \xi)$ be the probability density on E . The averaging operation mentioned in hypothesis 4' from section 2 is caused by the fluctuation process described by equation (1) where φ is the initial state and $\tilde{\varphi}$ is the asymptotic one as the time of fluctuation tends to infinity.

A corollary of this hypothesis: the map $\varphi \mapsto \tilde{\varphi}$ is obtained (by Theorem 3.1) as the composition of maps given by formulas (5) and (4) respectively.

Consider the Hilbert space $\tilde{\mathcal{H}}$ of real functions $\tilde{\varphi}$ on E of the form (4) and with the standard inner product

$$\langle \tilde{\varphi}', \tilde{\varphi}'' \rangle = \int_E \tilde{\varphi}' \tilde{\varphi}'' dq dp d\xi.$$

Denote by \mathcal{H} the Hilbert space of complex-valued functions $\psi(x, \xi)$ on $R^n \times F$ satisfying (2) and with the inner product given by the formula

$$\langle \psi_1, \psi_2 \rangle = \operatorname{Re} \int_{R^n \times F} \psi_1(x, \xi) \psi_2^*(x, \xi) dx d\xi.$$

Making use of Theorem 3.1 we directly derive the following

C o r o l l a r y 3.2. *The map given by formula (4) is an isomorphism of Hilbert spaces \mathcal{H} and $\tilde{\mathcal{H}}$.*

A function $\psi(x, \xi)$ satisfying (2) will be called a wave function, the corresponding function $\tilde{\varphi}(q, p, \xi)$ will be called the probability amplitude in the enlarged phase space.

R e m a r k. Formula (5) gives the so-called position representation of φ . The momentum representation of φ is given by the Fourier transform of $\psi(x, \xi)$ with respect to x . We will not need it.

4 An estimate of the operator of an observable

In this section we will estimate the operator \tilde{A}_f acting in the Hilbert space $\tilde{\mathcal{H}}$ of functions of the form (4). By definition \tilde{A}_f is given by the following expression:

$$\langle f, \tilde{\rho} \rangle_{\tilde{\mathcal{H}}} = \langle \tilde{\varphi}, A_f \tilde{\varphi} \rangle_{\tilde{\mathcal{H}}} = \int_E f(q, p) \tilde{\varphi}^2(q, p, \xi) dq dp d\xi. \quad (6)$$

Since by Corollary 3.2 the Hilbert spaces $\tilde{\mathcal{H}}$ and \mathcal{H} are isomorphic, we can estimate A_f by estimating image of \tilde{A}_f in \mathcal{H} under this isomorphism.

Let us substitute (4) in (6). Represent the function

$$\tilde{\varphi}^2(q, p, \xi) = \left(\int_{R^n} B(q, p, \xi, x) dx \right)^2,$$

where $B(q, p, \xi, x)$ is the integrand of (4), in the form

$$\tilde{\varphi}^2(q, p, \xi) = \int_{R^n} \int_{R^n} B(q, p, \xi, x') B(q, p, \xi, x) dx' dx.$$

Since the integrals of the product $\psi(x', \xi)\psi(x, \xi)$ and of the product $\psi^*(x', \xi)\psi^*(x, \xi)$ over ξ are zero due to (2), we get after simplification

$$\langle f, \tilde{\rho} \rangle_{\tilde{\mathcal{H}}} = \langle \tilde{\varphi}, A_f \tilde{\varphi} \rangle_{\tilde{\mathcal{H}}} = \langle \psi, A_f \psi \rangle_{\mathcal{H}} = \left(\frac{2}{h^3} \right)^{\frac{n}{2}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^{\frac{1}{2}} \times$$

$$\begin{aligned} & \times \int_{R^n} \int_{R^n} \int_E f(q, p) \exp \left[-\frac{\pi}{h} \sum_{i=1}^n \frac{b}{a} ((q_i - x'_i)^2 + (q_i - x_i)^2) \right] \times \\ & \times \psi(x', \xi) \psi^*(x, \xi) \exp \left(-j \frac{2\pi \langle p, x - x' \rangle}{h} \right) dq dp d\xi dx' dx. \end{aligned}$$

This implies that the kernel of A_f on the space \mathcal{H} is of the form

$$\begin{aligned} A_f(x, x') &= \left(\frac{2}{h^3} \right)^{\frac{n}{2}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^{\frac{1}{2}} \times \\ & \times \int_{R^{2n}} f(q, p) \exp \left[-\frac{\pi}{h} \sum_{i=1}^n \frac{b}{a} ((q_i - x'_i)^2 + (q_i - x_i)^2) \right] \times \\ & \times \exp \left(-j \frac{2\pi \langle p, x - x' \rangle}{h} \right) dq dp, \end{aligned} \quad (7)$$

and the density of the probability distribution $\tilde{\rho}(q, p) = \tilde{\varphi}^2(q, p)$ corresponding to $\psi(x, \xi)$ is given by the next theorem.

Theorem 4.1. *Let $\psi(x, \xi)$ be a wave function (in the position representation) on $R^n \times F$ satisfying (2); then the density of the density of the probability distribution on the phase space is given by the formula:*

$$\begin{aligned} \tilde{\rho}(q, p) &= \left(\frac{2}{h^3} \right)^{\frac{n}{2}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^{\frac{1}{2}} \times \\ & \times \int_F \int_{R^n} \int_{R^n} \exp \left[-\frac{\pi}{h} \sum_{i=1}^n \frac{b}{a} ((q_i - x'_i)^2 + (q_i - x_i)^2) \right] \times \\ & \times \exp \left(-j \frac{2\pi \langle p, x - x' \rangle}{h} \right) \psi(x', \xi) \psi^*(x, \xi) dx' dx d\xi. \end{aligned} \quad (8)$$

This distributions is different from Wigner's quasidistributions by the integration with the function

$$\left(\frac{2}{h} \right)^{\frac{n}{2}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^{\frac{1}{2}} \exp \left[-\frac{\pi}{h} \sum_{i=1}^n \frac{b}{a} ((q_i - x'_i)^2 + (q_i - x_i)^2) \right].$$

Now in order to represent the operator A_f in the form habitual in quantum mechanics (cf. [FYa]), substitute in (7) the expression of f in terms of

its Fourier transform:

$$f(q, p) = \left(\frac{1}{2\pi}\right)^n \int_{R^{2n}} \hat{f}(u, v) \exp[-j(\langle q, v \rangle + \langle p, u \rangle)] du dv,$$

where

$$\hat{f}(q, p) = \left(\frac{1}{2\pi}\right)^n \int_{R^{2n}} f(u, v) \exp[j(\langle q, v \rangle + \langle p, u \rangle)] dq dp.$$

In the obtained formula we first integrate over q , then over p and u . We get:

$$\begin{aligned} A_f(x, x') &= \frac{1}{h^n} \int_{R^{2n}} \hat{f}\left(\frac{2\pi(x - x')}{h}, v\right) \exp\left(-\frac{h}{8\pi} \sum_{i=1}^n \left(\frac{a_i}{b_i} v_i^2 + \right.\right. \\ &\quad \left.\left. + \frac{b_i}{a_i} \left(\frac{2\pi(x_i - x'_i)}{h}\right)^2\right)\right) \exp\left(-j \left\langle \frac{x + x'}{2}, v \right\rangle\right) dv = \\ &= \frac{1}{h^n} \int_{R^{2n}} \tilde{f}_h\left(\frac{2\pi(x - x')}{h}, v\right) \exp\left(-j \left\langle \frac{x + x'}{2}, v \right\rangle\right) dv, \end{aligned}$$

where

$$\tilde{f}_h(u, v) = \hat{f}(u, v) \exp\left(-\frac{h}{8\pi} \sum_{i=1}^n \left(\frac{a_i}{b_i} v_i^2 + \frac{b_i}{a_i} u_i^2\right)\right).$$

If instead of $\tilde{f}_h(u, v)$ we take its Taylor series expansion in powers of h up to order n we get the corresponding asymptotic representation of A_f . In particular, the asymptotics of the 0-th term of the expansion is

$$A_f(x, x') = \frac{1}{h^n} \int_{R^n} \hat{f}\left(\frac{2\pi(x - x')}{h}, v\right) \exp\left(-j \left\langle \frac{x + x'}{2}, v \right\rangle\right) dv.$$

This expression coincides with the expression for the operator of an observable in quantum mechanics (formula (14) in [FYa] whose h is our $h/2\pi$). This implies the next result of the paper:

Theorem 4.2. *The linear operator A_f of the form (7) in the Hilbert space \mathcal{H} which is constructed from the classical observable f under assumptions 1-5 on the process of quantum observation is asymptotically close to the conventional operator of quantum observable given in the position representation.*

If a more precise estimate of A_f is required we can always take into account more terms of the Taylor series expansion of $f_h(u, v)$ with respect to h .

Examples of exact formulas for A_f . By the usual abuse of language let us denote the operator of multiplication by a function F by F . We have:

$$\begin{aligned} A_{q_i} &= x_i; \\ A_{p_i} &= -j \frac{h}{2\pi} \frac{\partial}{\partial x_i}; \\ A_{q_i^2} &= x_i^2 + \frac{h a_i}{4\pi b_i}; \\ A_{p_i^2} &= -\frac{h^2}{4\pi^2} \frac{\partial^2}{\partial x_i^2} + \frac{h b_i}{4\pi a_i}. \end{aligned}$$

It follows that for the Hamiltonian $f = p^2/(2m) + m\omega^2 q^2/2$ of the linear oscillator with eigen frequency ω we have

$$A_f = -\frac{h^2}{8\pi^2 m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2} + \sum_{i=1}^3 \frac{h(b_i^2 + m^2\omega^2 a_i^2)}{8\pi a_i b_i m}.$$

which differs from the conventional operator of the quantum linear oscillator by constants (the last summand).

In other side, if $f = f(q)$, i.e. does not depend on momenta, we deduce from (7) after integration over p and x' that A_f is the operator of multiplication by

$$\bar{f}(x) = \left(\frac{2}{h}\right)^{\frac{n}{2}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(q) \exp\left(-\frac{2\pi}{h} \sum_{i=1}^n \frac{b_i}{a_i} (q_i - x_i)^2\right) dq.$$

The passage $f \mapsto \bar{f}$ is the convolution of f with the density of the probability distribution with dispersion along the q_i -axis equal to $h a_i / (4\pi b_i)$.

Suppose

$$a_1/b_1 = a_2/b_2 = a_3/b_3 = a/b.$$

Then for the Hamiltonian

$$f(q, p) = (p_1^2 + p_2^2 + p_3^2)/(2m) + V(q_1, q_2, q_3),$$

we have

$$A_f = -\frac{h^2}{8\pi^2m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) + \frac{3hb}{4\pi a} + \bar{V}(x_1, x_2, x_3). \quad (9)$$

In particular for the Hamiltonian of the hydrogen atom whose

$$V(q_1, q_2, q_3) = -e^2/r, \quad (10)$$

where e is the charge of the electron and $r^2 = \sum_{i=1}^3 q_i^2$, the operator A_f differs from the operator by an irrelevant constant $3hb/(4\pi a)$ and the extra smoothness of the Coulomb potential. Thus our hypotheses predict that the theoretical spectrum of the hydrogen atom computed on the base of the conventional Hamiltonian should differ from that obtained in experiments. Such a discrepancy of theory and experiment was indeed detected in 40s [LR]. It is called Lamb's shift of hydrogen atom's levels and is conventionally explained in quantum electrodynamics by an interaction of the electron with a fluctuating electromagnetic field. Comparison of these experimental data with calculations of the spectrum of A_f given by (9), (10) via perturbation theory yields the following estimate of parameters of our model (details see in Application A.2):

$$a/b = 3,41 \cdot 10^4 \text{ sec/gr}; \quad \Delta q = \sqrt{\frac{ha}{4\pi b}} = 4,24 \cdot 10^{-12} \text{ cm}.$$

Hence the standard deviation of the normal distribution Δq is comparable with $h/(2\pi mc) = 3,8 \cdot 10^{-11} \text{ cm}$, the minimal position error of electron (in the rest frame) obtained in quantum electrodynamics.

5 Conclusion and plan of further study

The description of quantum systems is usually based upon certain formal procedures starting from a classical description of the corresponding mechanical systems. Many physicists and mathematicians, starting with Einstein, searched for a meaning of these procedures but the success of quantum mechanics approved the formal approach to the quantization procedure and von Neumann's theorem on "hidden parameters" [Nu] discredited for a long time such a search as a direction of scientific investigations.

The interest to the problem was revived in 50s in works of Bohm and de Broglie [B], [Br] and maintained in a number of later works, cf. [BV], [N], [PG], [M], [Ba], [KV]. At the same time the rigidity and lack of motivation in the mathematical requirements in von Neumann's theorem became manifest [F].

A detailed analysis of the problem of introduction of hidden parameters in quantum mechanics and von Neumann's theorem is given in [Kh]. In particular, there is given a formal model with hidden parameters for a "solitary" quantum system. The difficulty of introducing the classical probabilistic model for quantum phenomena is associated with a non-local character of these phenomena, confirmed in a number of experiments, cf. the review [Gr], [SM] (however, cf. [B1], Appendix 1).

We have shown how to deduce operators of quantum observables on the base of hypotheses of sec. 1. The reason causing fluctuations on the extended phase space are diverse: a fluctuating external force, inaccuracy in the description of the real system, etc. We have shown that irrespective of the nature of the fluctuation quantum effects will be observed in such systems with accuracy determined by the Planck's constant and the ratios a_i/b_i , where a_i and b_i are the intensities of random shifts along the i -th position and momentum, respectively.

Elsewhere I intend to investigate the following problems:

- Take into the account relativistic effects. We have constructed A_f by formula (7) having given a classical observable f under the assumptions that the intensities of shifts a and b along the positions and momenta are constants. This is not a relativistic requirement and it is desirable to find the dependence of the intensities on momenta and the masses of particles and to refine formula (4).

- Take into the account spin. For this we should replace our group P by a supergroup whose Lie superalgebra is $\mathfrak{po}(2n|m)$ or its "odd" counterpart $\mathfrak{b}_\lambda(n)$, see [L].

- Generalize our construction to phase spaces more general than direct products of the configuration space by the space of momenta. This should lead to restrictions on the Planck's constant \hbar cases by the geometry of the phase space, cf. [B2].

- Describe dynamics of observable taking into account fluctuating action of the ambient media. As is shown in App.1 the time for stabilization (averaging) over fluctuations is of order $\hbar/(2\pi ab)$. If this quantity is small

the dynamics can be approximately described as the superposition of a fast and a slow movements. The fast motion leads to averaging of probability amplitudes and the slow one describes the motion in the space of averaged amplitudes. The classical Schroedinger equation only describes the slow constituent of the motion of a microobject.

Appendix 1. Proof of Theorem 3.1

Since (3.1) is determined via the action of the Heisenberg-Weyl group $W_t^{x,y}$ in the space of square integrable (and complexified for convenience) functions on E , let us decompose a function $\varphi(\tau, q, p, \xi)$ into the integral over irreducible representations of W .

The decomposition will be performed into three steps. First, let us consider the action of $T = R/hZ$ in the space of functions on F . Let $\varphi(\xi)$ be a function on F . Since $\varphi(T_t(x))$, where $T_t \in T$, is periodic in t with period h , it has the Fourier series expansion

$$\varphi(T_t(\xi)) = \sum_{k \in Z} \varphi_k(\xi) \exp\left(j \frac{2\pi kt}{h}\right),$$

where

$$\varphi_k(\xi) = \frac{1}{h} \int_t^h \varphi(T_t(\xi)) \exp\left(-j \frac{2\pi kt}{h}\right) dt.$$

In particular, for $t = 0$ we get $\varphi(\xi) = \sum_{k \in Z} \varphi_k(\xi)$. It follows from the definition of $\varphi_k(\xi)$ that $\varphi(\xi) \mapsto \varphi_k(\xi)$ is a projection and T acts on the functions $\varphi_k(\xi)$ by the formula

$$\varphi_k(T_t(\xi)) = \exp\left(j \frac{2\pi kt}{h}\right) \varphi_k(\xi). \quad (A.1)$$

As is easy to see, the functions φ_k and φ_n are orthogonal for $n \neq k$.

A function $\varphi(q, p, \xi)$ on the P -bundle $E = R^{2n} \times F$ can also be represented in the form

$$\varphi(q, p, \xi) = \sum_{k \in Z} \varphi_k(q, p, \xi), \quad (A.2)$$

where

$$\varphi_k(q, p, \xi) = \frac{1}{h} \int_t^h \varphi(q, p, T_t(\xi)) \exp\left(-j \frac{2\pi kt}{h}\right) dt.$$

Having substituted (A.2) into (3.1) and taking (A.1) into account we get the following equations for each of the orthogonal components of φ in coordinates on E given by Corollary 2.2

$$\begin{aligned} \varphi_k(\tau + \Delta\tau, q, p, \xi) &= \int_{R^{2n}} K(\Delta q, \Delta p, \Delta\tau) \varphi_k(\tau, q + \Delta q, p + \Delta p, \xi) \\ &\times \exp\left(-j \frac{2\pi k \langle \Delta p, q \rangle}{h}\right) d(\Delta q) d(\Delta p) + o(\Delta\tau). \end{aligned} \quad (A.3)$$

For $k = 0$ the equation (A.3) turns as $\Delta\tau \rightarrow 0$ (as is clear from the Taylor series expansion) into the following differential equation on $\varphi_0(\tau, q + \Delta q, p + \Delta p, \xi)$:

$$\frac{\partial \varphi_0}{\partial \tau} = \sum_{i=1}^n \left(a_i^2 \frac{\partial^2 \varphi_0}{\partial q_i^2} + b_i^2 \frac{\partial^2 \varphi_0}{\partial p_i^2} \right),$$

and since by hypothesis of Theorem 3.1

$$\varphi_0(0, q, p, \xi) = 1/h \int_t^h \varphi(q, p, T_t(\xi)) dt = 0,$$

then $\varphi_0(\tau, q, p, \xi) \equiv 0$ for any $\tau \geq 0$.

For $k \neq 0$ let us expand $\varphi_k(\tau, q, p, \xi)$ into the Fourier integral with respect to p , i.e. represent it in the form:

$$\varphi_k(\tau, q, p, \xi) = \left(\frac{|k|}{h} \right)^{\frac{n}{2}} \int_{R^n} \hat{\varphi}_k(\tau, q, x, \xi) \exp\left(j \frac{2\pi k \langle p, x \rangle}{h}\right) dx, \quad (A.4)$$

where

$$\hat{\varphi}_k(\tau, q, x, \xi) = \left(\frac{|k|}{h} \right)^{\frac{n}{2}} \int_{R^n} \varphi_k(\tau, q, p, \xi) \exp\left(-j \frac{2\pi k \langle p, x \rangle}{h}\right) dp. \quad (A.5)$$

R e m a r k. As is easy to verify, the functions of the form

$$\hat{\varphi}_k(\tau, q, x, \xi) \exp(j 2\pi k \langle p, x \rangle / h) \quad \text{for fixed } k, x, \xi \quad (A.6)$$

span a W -invariant subspace in the space of functions on E with the W -action described in the Corollary 2.2.

Since the spaces of functions (A.6) are orthogonal to each other for distinct k, x and are W -invariant, then equations (A.3) split into a system of

equations, each for each subspace, each obtained from (A.3) by substituting $\hat{\varphi}_k(\tau, q, x, \xi) \exp(j2\pi k \langle p, x \rangle / h)$.

As a result of all these transformations we get

$$\begin{aligned} \hat{\varphi}_k(\tau + \Delta\tau, q, x, \xi) &= \int_{R^{2n}} K(\Delta q, \Delta p, \Delta\tau) \hat{\varphi}_k(\tau, q + \Delta q, x, \xi) \times \\ &\times \exp\left(j \frac{2\pi k \langle \Delta p, x - q \rangle}{h}\right) d(\Delta q) d(\Delta p) + o(\Delta\tau), \end{aligned}$$

which by expansion of the function

$$\hat{\varphi}_k(\tau, q + \Delta q, x, \xi) \exp(j2\pi k \langle \Delta p, x - q \rangle / h)$$

into the Taylor series in powers of Δp and Δq and with the above properties of $K(\Delta q, \Delta p, \Delta\tau)$ take as $\Delta\tau \rightarrow 0$ the form

$$\frac{\partial \hat{\varphi}_k}{\partial \tau} = \sum_{i=1}^n \left(a_i^2 \frac{\partial^2 \hat{\varphi}_k}{\partial q_i^2} + b_i^2 \left(\frac{2\pi k}{h} (x_i - q_i) \right)^2 \hat{\varphi}_k \right), \quad (A.7)$$

where $\hat{\varphi}_k$ are functions in τ, q, x, ξ for $k \in Z \setminus \{0\}$.

To study eqs. (A.7), consider the corresponding eigenvalue problems:

$$a_i^2 \frac{\partial^2 \hat{\varphi}_k}{\partial q_i^2} + b_i^2 \left(\frac{2\pi k}{h} (x_i - q_i) \right)^2 \hat{\varphi}_k = \lambda \hat{\varphi}_k.$$

This is a stationary Schroedinger equation for harmonic oscillations [FYa]. If $\hat{\varphi}_k(q) \rightarrow 0$ as $|q| \rightarrow \infty$, the equation has a discrete spectrum. Its eigenvalues are of the form

$$\lambda_{k, k_1, \dots, k_n} = - \sum_{i=1}^n \frac{2\pi |k| a_i b_i}{h} (2k_i + 1), \quad (A.8)$$

where k_1, \dots, k_n are nonnegative integers. The corresponding eigen functions $\hat{\varphi}_{k, k_1, \dots, k_n}(q, x)$ are products of the Chebyshev-Hermit polynomials in $(b_i/a_i)\sqrt{2\pi|k|/h}(q_i - x_i)$ by $\exp(-\pi|k|/h \sum_{i=1}^n (b_i/a_i)(q_i - x_i)^2)$.

Since eigen functions of the Schroedinger equation are orthogonal to each other and constitute a complete system in the space of square integrable functions in q , then $\hat{\varphi}_k$ can be represented as the series:

$$\hat{\varphi}_k(\tau, q, x, \xi) = \frac{1}{\sqrt{2}} \sum_{k_1, \dots, k_n=0}^{\infty} c_{k, k_1, \dots, k_n}(\tau, x, \xi) \hat{\varphi}_{k, k_1, \dots, k_n}(q, x), \quad (A.9)$$

where

$$c_{k,k_1,\dots,k_n}(\tau, x, \xi) = \sqrt{2} \int_{\mathbb{R}^n} \hat{\varphi}_k(\tau, q, x, \xi) \hat{\varphi}_{k,k_1,\dots,k_n}(q, x) dq.$$

The multiple $1/\sqrt{2}$ is taken for convenience. In particular, the maximal eigen value $\lambda_{\pm 1,0,\dots,0} = -\sum_{i=1}^n 2\pi a_i b_i/h$ is attained on the normed eigen function

$$\hat{\varphi}_{\pm 1,0,\dots,0} = \left(\frac{2}{h}\right)^{\frac{n}{4}} \left(\frac{b_1\dots b_n}{a_1\dots a_n}\right)^{\frac{1}{4}} \exp\left(-\frac{\pi}{h} \sum_{i=1}^n \frac{b_i}{a_i} (q_i - x_i)^2\right), \quad (\text{A.10})$$

where

$$c_{\pm 1,0,\dots,0} = \sqrt{2} \left(\frac{2}{h}\right)^{\frac{n}{4}} \left(\frac{b_1\dots b_n}{a_1\dots a_n}\right)^{\frac{1}{4}} \int_{\mathbb{R}^n} \hat{\varphi}_{\pm 1} \exp\left(-\frac{\pi}{h} \sum_{i=1}^n \frac{b_i}{a_i} (q_i - x_i)^2\right) dq.$$

Now, let us return to the nonstationary equations (A.7). Having substituted in them the expressions for $\hat{\varphi}_{k,k_1,\dots,k_n}(q, x)$ are eigen functions of the right hand side of the equation, we get equations

$$\sum_{k_1,\dots,k_n=0}^{\infty} \frac{\partial c_{k,k_1,\dots,k_n}}{\partial \tau} \hat{\varphi}_{k,k_1,\dots,k_n} = \sum_{k_1,\dots,k_n=0}^{\infty} \lambda_{k,k_1,\dots,k_n} c_{k,k_1,\dots,k_n} \hat{\varphi}_{k,k_1,\dots,k_n},$$

which due to orthogonality of the functions $\hat{\varphi}_{k,k_1,\dots,k_n}(q, x)$ in the space of square integrable over q functions split into the system of equations

$$\frac{\partial c_{k,k_1,\dots,k_n}}{\partial \tau} = \lambda_{k,k_1,\dots,k_n} c_{k,k_1,\dots,k_n}$$

for $k \in Z \setminus \{0\}$, $k_1 \geq 0, \dots, k_n \geq 0$. The solutions of these equations are of the form

$$c_{k,k_1,\dots,k_n}(\tau, x, \xi) = c_{k,k_1,\dots,k_n}(0, x, \xi) \exp(\lambda_{k,k_1,\dots,k_n} \tau). \quad (\text{A.11})$$

Since $\lambda_{k,k_1,\dots,k_n} < 0$, then c_{k,k_1,\dots,k_n} decrease exponentially as $\tau \rightarrow \infty$ and the largest contribution to $\hat{\varphi}_k(\tau, q, x, \xi)$ is given for not too small τ by the terms of the series (A.9) with the largest eigen value.

Thus, with (A.9), (A.11), (A.8) we get the following asymptotic in $\tau \rightarrow \infty$:

$$\hat{\varphi}_k(\tau, q, x, \xi) \sim \frac{1}{\sqrt{2}} c_{k,0,\dots,0}(0, x, \xi) \hat{\varphi}_{k,0,\dots,0}(q, x) \exp\left(-\tau \sum_{i=1}^n \frac{2\pi |k| a_i b_i}{h}\right).$$

With (A.4) we deduce that in (A.2) every summand φ_k with $k \neq 0$ exponentially decrease with the growth of τ . Hence the largest contribution to φ is given after a while by the terms with the largest exponent, i.e. for $k = \pm 1$. In other words, we have, asymptotically,

$$\varphi \sim \varphi_{-1} + \varphi_1.$$

Having substituted here, consecutively, (A.4) for $k = \pm 1$, the asymptotic expressions for $\hat{\varphi}_{\pm 1}$ obtained above and expressions (A.10) for $\hat{\varphi}_{\pm 1, 0, \dots, 0}$ we finally get

$$\begin{aligned} \varphi(\tau, q, p, \xi) \sim & \frac{1}{\sqrt{2}} \left(\frac{2}{h^3} \right)^{\frac{n}{4}} \left(\frac{b_1 \dots b_n}{a_1 \dots a_n} \right)^{\frac{1}{4}} \exp \left(-\tau \sum_{i=1}^n \frac{2\pi |k| a_i b_i}{h} \right) \times \\ & \times \int_{R^n} \exp \left(-\frac{\pi}{h} \sum_{i=1}^n \frac{b_i}{a_i} (q_i - x_i)^2 \right) \times \\ & \times \left(c_{-1, 0, \dots, 0}(0, x, \xi) \exp \left(-j \frac{2\pi k \langle p, x \rangle}{h} \right) + \right. \\ & \left. + c_{1, 0, \dots, 0}(0, x, \xi) \exp \left(j \frac{2\pi k \langle p, x \rangle}{h} \right) \right) dx, \end{aligned} \quad (\text{A.12})$$

where $c_{\pm 1, 0, \dots, 0}(0, x, \xi)$ is the result of operations

$$\varphi(q, p, \xi) = \varphi(0, q, p, \xi) \mapsto \varphi_{\pm 1}(q, p, \xi) \mapsto \hat{\varphi}_{\pm 1}(q, p, \xi) \mapsto c_{\pm 1, 0, \dots, 0}(0, x, \xi)$$

according to the formulas (A.2), (A.5), (A.10).

It is not difficult to verify that if φ is a real function then $c_{1, 0, \dots, 0}(0, x, \xi) = c_{-1, 0, \dots, 0}^*(0, x, \xi)$. Therefore, setting

$$\psi(x, \xi) = c_{-1, 0, \dots, 0}(0, x, \xi)$$

and having substituted this into (A.12) we get Theorem 3.1.

Appendix 2. Estimate of the parameter a/b of the considered model

In this estimate we follow the method of [We] to justify the Lamb's shift. Consider the operator A_f given by (9) for the Hamiltonian $f(q, p)$ as a perturbation of the operator H for the hydrogen atom of the quantum mechanics. The perturbation theory implies that in the first order the increment δE_n of the eigenvalue E_n of H is of the form

$$\delta E_n = \int_{R^3} \rho_n(x) (\bar{V}(x) - V(x)) dx,$$

where $\rho_n(x) = |\psi_n(x)|^2$ and $\psi_n(x)$ is the eigenfunction of with eigenvalue E_n and where $V(x) = -e^2/\sqrt{x_1^2 + x_2^2 + x_3^2} = -e^2/r$. By definition the function $\bar{V}(x)$ is the mathematical expectation $\bar{V}(x) = M_q V(x + q)$, where q is normally distributed with density

$$\left(\frac{2b}{ha}\right)^{\frac{3}{2}} \exp\left(-\frac{2\pi b}{ha}(q_1^2 + q_2^2 + q_3^2)\right).$$

After simplification we get:

$$\delta E_n = \int_{R^3} (M_q(\rho_n(x + q)) - \rho_n(x)) V(x) dx.$$

Since $\rho_n(x)$ is smooth (infinitely differentiable), then if the standard deviation of q is essentially smaller than the atom's radius we can make use of the Taylor series expansion of $\rho_n(x + q)$ to compute $M_q(\rho_n(x + q))$. We have

$$\rho_n(x + q) \approx (1 + \langle q, \nabla \rangle + \frac{1}{2} \langle q, \nabla \rangle^2) \rho_n(x).$$

Since $M_q(q_i) = 0$, $M_q(q_i q_j) = 0$ for $i \neq j$ and $M_q(q_i^2) = ha/(4\pi b)$, then we get the approximate equality

$$M_q(\rho_n(x + q)) = \rho_n(x) + \frac{ha}{8\pi b} \Delta \rho_n(x),$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$.

Having substituted this $M_q(\rho_n(x+q))$ into the expression for δE_n and since Δ is self adjoint, we get

$$\delta E_n = \frac{ah}{8\pi b} \int_{R^3} \rho_n(x) \sum_{k=1}^3 \frac{\partial^2 V(x)}{\partial x_k^2} dx.$$

Since for $V(x) = -e^2/r$ we have $\Delta V(x) = 4\pi e^2 \delta_0(x)$, where $\delta_0(x)$ is Dirac delta-function,

$$\delta E_n = \frac{ah e^2}{2b} \rho_n(0).$$

For the hydrogen atom

$$\rho_n(0) = |\psi_n(0)|^2 = \frac{1}{\pi n^3} \left(\frac{m e^2}{\hbar^2} \right)^3,$$

see e.g. [STZh] p.342, where $\hbar = h/(2\pi)$, and therefore

$$\delta E_n = \frac{a \hbar \pi e^2}{b} \frac{1}{\pi n^3} \left(\frac{m e^2}{\hbar^2} \right)^3 = \frac{a}{b} \frac{m^3 e^8}{n^3 \hbar^5} = \frac{a}{b} \frac{m^3 \alpha^4 c^4}{n^3 \hbar},$$

where $\alpha = e^2/(\hbar c) = 1/137$ and c is the speed of light in vacuum. It follows

$$\frac{a}{b} = \delta E_n \frac{n^3 \hbar}{m^3 \alpha^4 c^4}.$$

In experiments of Lamb and Retherford [LR] it had been established that for the hydrogen atom $\delta E_2 = 1058 gHz = 1058 \cdot 10^6 h$ erg. Comparing this expression with our value of δE_2 , we directly get the estimate for a/b :

$$a/b = 3,41 \cdot 10^4 sec/gr.$$

Accordingly, the standard deviation of q in each coordinate is

$$\Delta q_i = \sqrt{a \hbar / 2b} = 4,24 \cdot 10^{-12} cm,$$

which is essentially smaller than the radius of the hydrogen atom.

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